## §5.4 Instantons

Consider the path integral on a space with Euclidean metric (Wick rotate otherwise)

→ have to find local minima of action and evaluate quantum fluctuations around them

→ consider SU(2) gauge theory on Euclidean $\mathbb{R}^4$

→ local minima of

$$S[A] = \int d^4x \, \frac{1}{2g^2} \, \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2}\int \mathrm{tr}\,(\mathcal{F} \wedge * \mathcal{F}) \quad (1)$$

are known as "instantons"

eq. of motion from (1): $D_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0$

→ given by configurations where

$$A_\mu \longrightarrow U(x)^{-1} \partial_\mu U(x) \quad \text{as} \quad |x| \longrightarrow \infty \quad (2)$$

because

$$\begin{aligned} F_{\mu\nu} \rightarrow \; & \partial_\mu [U^{-1}\partial_\nu U] - \partial_\nu [U^{-1}\partial_\mu U] \\ & + [U^{-1}\partial_\mu U, U^{-1}\partial_\nu U] \\ = \; & (\partial_\mu U^{-1})(\partial_\nu U) - (\partial_\nu U^{-1})(\partial_\mu U) \\ & + \left(-(\partial_\mu U^{-1})\partial_\nu U + (\partial_\nu U^{-1})\partial_\mu U\right) \\ = \; & 0 \end{aligned}$$

## The (anti-) self-dual solution

Let us consider the inequality

$$\int d^4x \; tr\left(F_{\mu\nu} \mp {*F_{\mu\nu}}\right)^2 \geq 0 \qquad (*)$$

$\longrightarrow$ saturated if

$$F_{\mu\nu} = \pm {*F_{\mu\nu}} \qquad (3)$$

where ${*F_{\mu\nu}} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho} F_{\lambda\rho}$ in Euclidean space

Then

$$D_\mu {*F_{\mu\nu}} = \partial_\mu {*F_{\mu\nu}} + [A_\mu, {*F_{\mu\nu}}]$$

$$= \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}\Big(2\partial_\mu\partial_\lambda A_\rho + \underbrace{\partial_\mu [A_\lambda, A_\rho]}_{\substack{= -[\partial_\rho A_\lambda, A_\mu] \\ -[A_\mu, \partial_\lambda A_\rho]}} + 2[A_\mu, \partial_\lambda A_\rho]$$

$$+ [A_\mu, [A_\lambda, A_\rho]]\Big)$$

$$= 0$$

$$\overset{(3)}{\longrightarrow} D_\mu F_{\mu\nu} = \pm D_\mu {*F_{\mu\nu}} = 0$$

eqs. of motion are satisfied!

Now use

$$S[A] = \frac{1}{2g^2}\int d^4x \; tr\, F_{\mu\nu}F^{\mu\nu}$$

$$= \frac{1}{4g^2}\int d^4x \; tr\left(F_{\mu\nu} \mp {*F^{\mu\nu}}\right)^2 \pm 2 tr\, F_{\mu\nu} {*F^{\mu\nu}}$$

$$\overset{(*)}{\geq} \pm\frac{1}{2g^2}\int d^3x \; \partial_\mu\left(\partial_\nu F_{\rho\sigma} + \frac{2}{3}A_\nu A_\rho A_\sigma\right)\varepsilon^{\mu\nu\rho\sigma} \qquad (4)$$

Solution (3) makes the YM-action to satisfy inequality $\longrightarrow$ surface term (will see is topological)

in diff. form notation:

$$S_{YM}[\mathcal{A}] = -\frac{1}{2}\int_M tr\left(\mathcal{F}\wedge *\mathcal{F}\right) \overset{(3)}{=} \mp\frac{1}{2}\int_M tr\left(\mathcal{F}\wedge\mathcal{F}\right)$$

impose eq. (2) (finiteness of action):

$$U : S^3_{R=L} \longrightarrow SU(2)$$

classified by $\pi_3(SU(2)) \cong \mathbb{Z}$

$\longrightarrow$ compactify $\mathbb{R}^4$ by adding $\{\infty\} \to S^4$ and cover by charts

$$U_N = \left\{ x \in \mathbb{R}^4 \,\middle|\, |x| \leq L + \varepsilon \right\}$$

$$U_S = \left\{ x \in \mathbb{R}^4 \,\middle|\, |x| \geq L - \varepsilon \right\}$$

$\longrightarrow$ choose $\mathcal{A}_S(x) = 0$ , $x \in U_S$

then for $x \in U_N \cap U_S$ :

$$\mathcal{A}_N = t_{NS}^{-1}\mathcal{A}_S t_{NS} + t_{NS}^{-1} d t_{NS} = t_{NS}^{-1} d t_{NS}$$

$\longrightarrow$ $U = t_{NS} : S^3 \longrightarrow SU(2)$

Let us compute the degree of this map!

Note that $SU(2) \simeq S^3$ since

$$t^4 \mathbb{1}_2 + i t^k \sigma_k \in SU(2) \iff \vec{t}^{\,2} + (t^4)^2 = 1$$

$$\rightarrow \quad \pi_3(SU(2)) \simeq \pi_3(S^3) = \mathbb{Z}$$

We have

i) The constant map

$$U_0 : x \in S^3 \mapsto \mathbb{1}_2 \in SU(2)$$

$\rightarrow$ belongs to $0 \in \pi_3(SU(2))$ (no winding)

ii) The "identity" map

$$U_1 : x \mapsto \frac{1}{r}\left[x^4 \mathbb{1}_2 + i x^k \sigma_k\right], \quad r^2 = \vec{x}^{\,2} + (x^4)^2$$

defines the class $1 \in \pi_3(SU(2))$

iii) The map

$$U_n := (U_1)^n : x \mapsto r^{-n}\left[x^4 \mathbb{1}_2 + i x^k \sigma_k\right]^n$$

defines class $n \in \pi_3(SU(2))$

Let us now evaluate the surface term given in (4):

note that $d \, \mathrm{tr}\, \mathcal{F} \wedge \mathcal{F} = \mathrm{tr}\left[d\mathcal{F} \wedge \mathcal{F} + \mathcal{F} \wedge d\mathcal{F}\right]$

$$= \mathrm{tr}\left\{-[\mathcal{A}, \mathcal{F}] \wedge \mathcal{F} - \mathcal{F} \wedge [\mathcal{A}, \mathcal{F}]\right\} = 0$$

where we used $\mathcal{D}\mathcal{F} = d\mathcal{F} + [\mathcal{A}, \mathcal{F}] = 0$

$\longrightarrow$ $\mathrm{tr}\,\mathcal{F}^2$ is <u>locally</u> exact:

$$\mathrm{tr}\,\mathcal{F}^2 = dK$$

Mathematically: $\operatorname{tr} \mathcal{F}^2 = Ch_2$
(second Chern character)
and $K$ is "Chern-Simons" form

<u>Lemma</u>:
The 3-form $K$ is given by
$$K = \operatorname{tr}\left[\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right]$$

<u>Proof</u>:
$$dK = \operatorname{tr}\left[(d\mathcal{A})^2 + \frac{2}{3}\left(d\mathcal{A}\mathcal{A}^2 - \mathcal{A}d\mathcal{A}\mathcal{A} + \mathcal{A}^2 d\mathcal{A}\right)\right]$$
$$= \operatorname{tr}\Big[(\mathcal{F}-\mathcal{A}^2)(\mathcal{F}-\mathcal{A}^2) + \frac{2}{3}\left((\mathcal{F}-\mathcal{A}^2)\mathcal{A}^2 - \mathcal{A}(\mathcal{F}-\mathcal{A}^2)\mathcal{A} + \mathcal{A}^2(\mathcal{F}-\mathcal{A}^2)\right)\Big]$$
$$= \operatorname{tr}\Big[\mathcal{F}^2 - \mathcal{A}^2\mathcal{F} - \mathcal{F}\mathcal{A}^2 + \mathcal{A}^4 + \frac{2}{3}\left(\mathcal{F}\mathcal{A}^2 - \mathcal{A}\mathcal{F}\mathcal{A} + \mathcal{A}^2\mathcal{F} - \mathcal{A}^4\right)\Big]$$

note that $\operatorname{tr}\mathcal{A}^4 = 0$, $\operatorname{tr}\mathcal{A}\mathcal{F}\mathcal{A} = -\operatorname{tr}\mathcal{A}^2\mathcal{F} = -\operatorname{tr}\mathcal{F}\mathcal{A}^2$
(use cyclicity of trace and anti-commutativity of $dx^\mu$)

$\rightarrow dK = \operatorname{tr}\mathcal{F}^2$

$\square$

Lemm 2:

Let $\mathcal{A}$ be the gauge potential of an instanton. Then

$$\int_{S^4} tr\, \mathcal{F}^2 = -\frac{1}{3}\int_{S^3} tr\, \mathcal{A}^3$$

Proof:

From Stokes's theorem, we find that

$$\int_{U_N} tr\, \mathcal{F}^2 = \int_{U_N} dK = \int_{S^3} K$$

where we used $S^3 = \partial U_N$. We have

$$\mathcal{F}\Big|_{S^3_{R=L}} = 0$$

$$\longrightarrow K = tr\left[\mathcal{A}\, d\mathcal{A} + \frac{2}{3}\mathcal{A}^3\right] = tr\left[\mathcal{A}(\mathcal{F}-\mathcal{A}^2) + \frac{2}{3}\mathcal{A}^3\right]$$

$$= -\frac{1}{3}\, tr\, \mathcal{A}^3$$

giving $\displaystyle\int_{U_N} tr\, \mathcal{F}^2 = \int_{S^4} tr\, \mathcal{F}^2 = -\frac{1}{3}\int_{S^3} tr\, \mathcal{A}^3$

where we used $\mathcal{A}_S = 0$ □

Note: under gauge trfs.

$$tr\, \mathcal{F}^2 \longrightarrow tr\left[U^{-1}\mathcal{F}^2 U\right] = tr\, \mathcal{F}^2$$

We have the following:

i) For $U_0(x) = \mathbb{1} \in SU(2) \rightarrow \mathcal{A} = 0$ on $S^3$

$$\rightarrow \int_{S^4} \operatorname{tr} \mathcal{F}^2 = -\frac{1}{3} \int_{S^3} \mathcal{A}^3 = 0$$

ii) Consider gauge potential with boundary values on $S^3$ given by

$$\mathcal{A} = \frac{1}{r}\left(x^4 - i x^k \sigma_\ell\right) d\left(\frac{1}{r}(x^4 + i x^\ell \sigma_\ell)\right)$$

evaluating at the north pole $(x^4 = 1, \vec{x} = 0)$ gives $\mathcal{A} = i\sigma_k dx^k$

$$\rightarrow \operatorname{tr} \mathcal{A}^3 = i^3 \operatorname{tr}[\sigma_i \sigma_j \sigma_k]\, dx^i \wedge dx^j \wedge dx^k$$

$$= 2 \varepsilon_{ijk}\, dx^i \wedge dx^j \wedge dx^k$$

$$= 12 \underbrace{dx^1 \wedge dx^2 \wedge dx^3}_{=: \omega}$$

volume form (at north pole)

$$\rightarrow \int_{S^3} \operatorname{tr} \mathcal{A}^3 = 12 \int_{S^3} \omega = 12(2\pi^2) = 24\pi^2$$

where $2\pi^2$ is the area of the unit sphere $S^3$

$$\rightarrow -\frac{1}{8\pi^2} \int_{S^3} \operatorname{tr} \mathcal{F}^2 = \frac{1}{24\pi^2} \int_{S^3} \operatorname{tr} \mathcal{A}^3 = 1$$

iii) Next, consider the map $U_n : S^3 \to SU(2)$ given by

$$x \mapsto r^{-n}\left[x^4 \mathbb{1}_2 + i x^k \sigma_k\right]^n$$

we show that $U_2 = U_1 U_1$ has winding number 2:

Let $S^3$ be covered by $U_N^{(3)}$ and $U_S^{(3)}$

for $U_1 : S^3 \to SU(2)$ deform to obtain

$$U_{1N}(x) = \mathbb{1} \quad \text{for } x \in U_S^{(3)}$$

$$\text{or } U_{1S}(x) = \mathbb{1} \quad \text{for } x \in U_N^{(3)}$$

$$\to \quad U_2'(x) = \begin{cases} U_{1N}(x) & x \in U_N^{(3)} \\ U_{1S}(x) & x \in U_S^{(3)} \end{cases}$$

For $\mathcal{A}(x) = U_2'(x)^{-1} dU_2'(x)$ $(x \in S^3)$, we have

$$\frac{1}{24\pi^3}\int_{S^3} tr\,\mathcal{A}^3 = \frac{1}{24\pi^2}\left(\int_{U_N^{(3)}} tr\left(U_{1N}^{-1} d\,U_{1N}\right)^3 + \int_{U_S^{(3)}} tr\left(U_{1S}^{-1} dU_{1S}\right)^3\right)$$

$$= 1 + 1 = 2$$

Repeating, we find for $\mathcal{A}(x) = U_n^{-1} dU_n$:

$$-\frac{1}{8\pi^2}\int_{S^4} tr\,\mathcal{F}^2 = \frac{1}{24\pi^2}\int_{S^3} tr\,\mathcal{A}^3 = n$$